6–VALENT ANALOGUES OF EBERHARD'S THEOREM†

BY

JOSEPH ZAKS

ABSTRACT

It is shown that for every sequence of non-negative integers $(p_n \mid 1 \leqq n \neq 3)$ satisfying the equation $\sum_{n\geqq 1}(3-n)p_n = 6$ (respectively, $= 0$) there exists a 6-valent, planar (toroidal, respectively) multi-graph that has precisely p_n n gonal faces for all $n, 1 \leqq n \neq 3$. This extends Eberhard's theorem that deals, in a similar fashion, with 3-valent, 3-connected planar graphs; the equation involved follows from the famous Euler's equation.

If G is a k-valent, connected planar graph possibly with multiple edges and loops, and G has precisely p_n n-gons for all $n \geqq 1$, then it follows by Euler's formula that

$$\sum_{n\geqq 1} (2k+2n-nk)p_n = 4k. \tag{1}$$

In case $k = 3$, equation (1) becomes

$$\sum_{n\geqq 1} (6-n)p_n = 12, \tag{2}$$

and in case $k = 6$, we have

$$\sum_{n\geqq 1} (3-n)p_n = 6. \tag{3}$$

The corresponding equations for toroidal graphs (that is, graphs which are 2-cell embedded in the torus) are:

$$\sum_{n\geqq 1} (6-n)p_n = 0, \text{ if } k = 3, \text{ and} \tag{4}$$

† This research was supported in part by the Office of Naval Research Grant N00014–67–0103–0003.

Received September 27, 1973

$$(5) \qquad \sum_{n \geqq 1} (3-n)p_n = 0, \text{ if } k = 6.$$

Observe that p_6 is not involved in equations (2) and (4), while p_3 is not involved in equations (3) and (5). The sequence $(p_n \mid n \geqq 1)$ is called the *p-vector* of the graph G.

A sequence $(p_6 \mid 1 \leqq n \neq 6)$ of non-negative integers that satisfies equation (2) (equation (4)) is called *3-realisable* (*toroidal 3-realizable*, respectively) if there exists a value for p_6 and a 3-valent, connected planar graph (toroidal graph, respectively) that has precisely p_n n-gons for all $n \geqq 1$. If $p_1 = 0$, then the graph is required to be 2-connected, and if $p_1 = p_2 = 0$ then the graph has to be 3-connected.

Eberhard's theorem ([3], see also [4]) states that every sequence $(p_n \mid p_1 = p_2 = 0, 1 \leqq n \neq 6)$ of non-negative integers that satisfies equation (2) is 3-realizable. Jendroľ and Jucovič [9] proved that a sequence $(p_n \mid p_1 = p_2 = 0, 1 \leqq n \neq 6)$ of non-negative integers that satisfies equation (4) is toroidal 3-realizable if and only if it differs from the sequence $(p_5 = p_7 = 1, p_n = 0$ for all $n \neq 5,6,7)$, (See also [5] and [7].) For similar results on 4-realizability, with $p_1 = p_2 = 0$, see [6] in the planar case and [13] and [1] for the toroidal case. Rowland [12] extended Eberhard's theorem concerning planar 3- and 4-realizabilities in cases where p_1 and p_2 are not necessarily both equal to zero.

Recently, Grünbaum and Zaks [8] established the following result which settles a conjecture of Brunel (1898–9, [2]) and, independently, of Malkevitch (1970, [10]).

Result 1. The sequence $(p_2 = 3, p_n = 0$ for all $n \neq 2,6)$ is the p-vector of a 3-valent, planar, 2-connected graph if and only if p_6 is of the form $x^2 + xy + y^2 - 1$, where x and y are non-negative integers and $(x,y) \neq (0,0)$.

The purpose of this paper is to introduce and study the notion of 6-realizability defined as follows: a sequence $(p_n \mid 1 \leqq n \neq 3)$ of non-negative integers is called *6-realizable* (*toroidal 6-realizable*) if there exists a value for p_3 and a 6-valent, connected planar graph (toroidal graph, respectively) that has precisely p_n n-gons for all $n \geqq 1$.

Another result of Grünbaum and Zaks [8] is the following.

Result 2. The sequence $(p_1 = 3, p_n = 0$ for all $n \neq 1,3)$ is the p-vector of a 6-valent, connected planar graph if and only if p_3 is of the form $2(x^2 + xy + y^2) - 1$, where x and y are non-negative integers and $(x,y) \neq (0,0)$.

We will establish here the following two 6-valent analogues of Eberhard's theorem.

THEOREM 1. *Every sequence* $(p_n \mid 1 \leqq n \neq 3)$ *of non-negative integers that satisfies equation* (3) *is 6-realizable.*

THEOREM 2. *Every sequence* $(p_n \mid 1 \leqq n \neq 3)$ *of non-negative integers that satisfies equation* (5) *is toroidal 6-realizable.*

Our proofs lean heavily on [8] in which the complete structure of 6-valent planar graphs G is determined for graphs G which have p-vectors of the form: $(p_1 = 3,\ p_n = 0$ for all $n \neq 2, 3)$. These graphs consists of three k-patches and closing triangles, where a *k-patch* is that part of the 6-valent, planar infinite graph H (described in Figure 1) which is included in the k concentric inner circles around the monogon; Figure 2 shows a 2-patch, while the part of H shown in Figure 1 is actually a 5-patch.

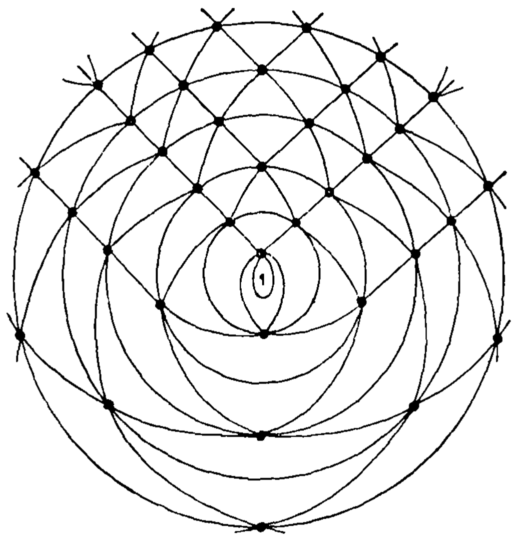

Fig. 1

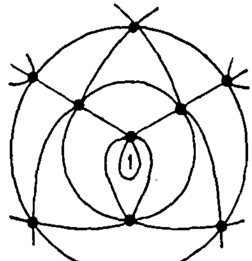

Fig. 2

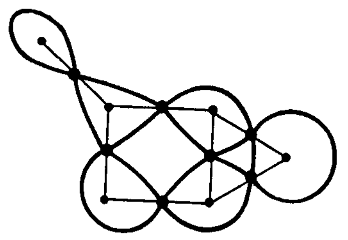

Fig. 3.

Let G be a graph in the plane (or in the torus) T; the *medial graph* of G (see [11, p. 47]), to be denoted by $M(G)$, is defined as follows: its vertices are interior points of the edges of G, one vertex on every edge of G; two vertices of $M(G)$ form an edge (of $M(G)$) if their corresponding edges of G are consecutive edges on a face of G. Figure 3 shows a graph G and its medial graph $M(G)$ (heavy line).

Clearly, $M(G)$ is a 4-valent graph and every n-valent vertex of G corresponds to an n-gon of $M(G)$. Let $S(G) = G \cup M(G)$ denote the union of the graph $M(G)$ with the subdivision graph of G, obtained from G by adding the vertices of $M(G)$. It follows immediately that if G is a connected 6-valent graph in T, then $S(G)$ is a connected 6-valent graph in T, and the p-vectors of G and $S(G)$ differ only in the number of triangles; an n-gon in G corresponds to an n-gon surrounded by triangles in $S(G)$.

Define *Operation* 1 of cactii-growing as follows†: a monogon inside an n-gon, $n \geqq 2$, is converted into two monogons, a triangle and an $(n+2)$-gon, such that one monogon is inside the triangle and the other monogon is inside the $(n+2)$-gon (so as to allow repetitions of the operation), as shown in Figure 4.

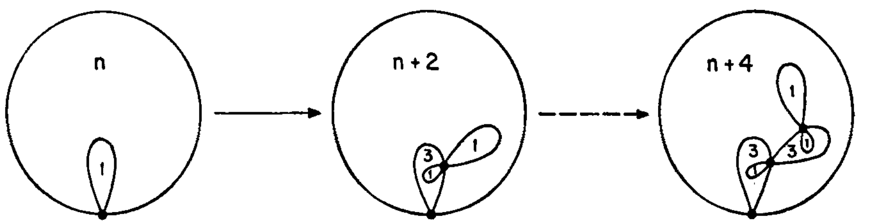

Fig. 4.

We need the following lemma.

† In Israel this is called sabra-growing.

Lemma. *If the sequence* $(p_n \mid 1 \leqq n \neq 3)$ *is 6-realizable in the plane (torus), and* $p_1 \geqq 1$, *then every sequence* $(q_n \mid 1 \leqq n \neq 3)$ *of non-negative integers satisfying equation* (3) (*equation* (5), *respectively*) *is 6-realizable in the plane (torus, respectively), provided*

i. $q_n = p_n$ *for all even* n *and*

ii. $q_n \geqq p_n$ *for all odd* n, $n \geqq 5$.

Proof. Let G be a 6-valent connected graph in the plane (torus), having the p-vector $(p_n \mid n \geqq 1)$ for some value of p_3. The graph G_1, taken as $S(G)$, has the same p-vector as that of G, except for the number of triangles, and its monogons are each inside triangles. Let G_2 be the graph obtained from G_1 by applying Operation 1 to one monogon of G_1 and repeating it k times. G_2 is 6-valent and it has p_{m+1} m-gons for $m = 3 + 2k$ and p_n n-gons for all n, $4 \leqq n \neq m$ or $n = 2$. This process, when applied for every m for which $q_m > p_m$, yields a 6-realization of the sequence $(q_n \mid 1 \leqq n \neq 3)$, as promised by the Lemma.

Let *Operation j*, for $2 \leqq j \leqq 6$, be the replacement as shown in Figures 5–9.

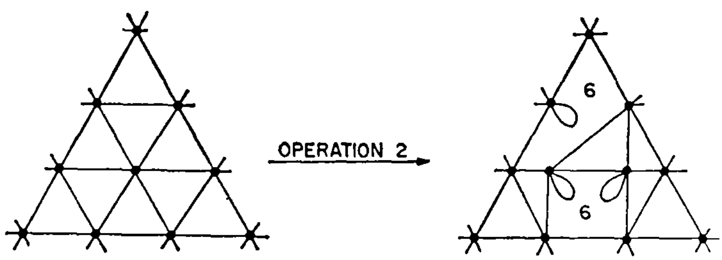

Fig. 5

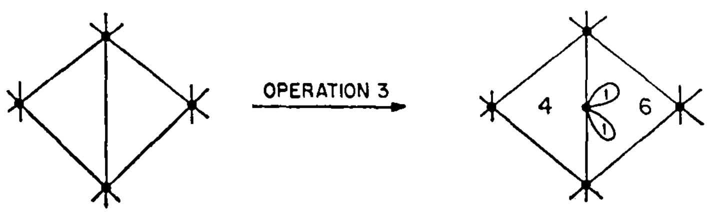

Fig. 6

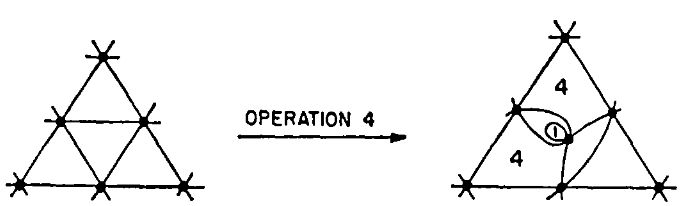

Fig. 7

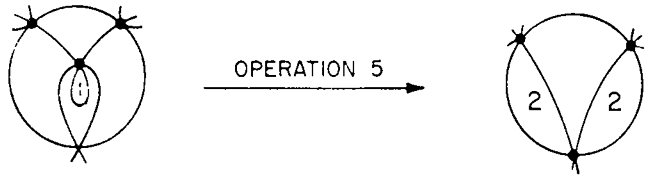

Fig. 8

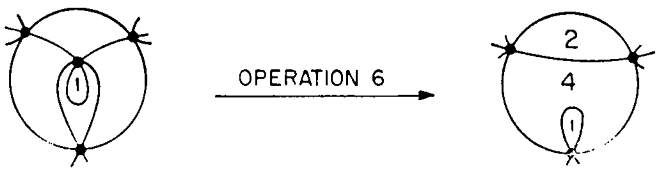

Fig. 9

NOTE. To simplify notation for publication purposes we use $\Sigma(p_n \mid n \in N)$ instead the usual $\Sigma_{n \in N} p_n$, where N is a set of natural numbers and p_n is an integer for every n in N.

PROOF OF THEOREM 1. Let $(p_n \mid 1 \leqq n \neq 3)$ be an arbitrary sequence of non-negative integers satisfying equation (3), that is,

$$2p_1 + p_2 = 6 + \sum_{n \geqq 4} (n-3)p_n.$$

Case I. $p_2 = 2k$ is even, $k \geqq 0$. Let the new sequence $(p_n' \mid 1 \leqq n \neq 3)$ be defined by $p_1' = p_1 + k$, $p_2' = 0$ and $p_n' = p_n$ for all $n \geqq 4$. We will first 6-realize this new sequence, then convert $\frac{1}{2}p_2$ of its monogons into p_2 digons plus triangles.

Let G^* be a 6-valent, connected planar graph with three monogons and t triangles, given by the above-mentioned Result 2 of [8], where t is so big as to allow all the needed operations in a non-interfering manner.

A pair of a $(6+2n)$-gon and a $(6+2m)$-gon, together with $3+n+m$ monogons and triangles is obtained by Operation 2, followed by repeating Operation 1 on the monogon inside a k-gon, for $k \geqq 6$. If $\Sigma(p_n \mid n$ even, $n \geqq 6)$ is even, all the needed n-gons, for even n, $n \geqq 6$, have been inserted. If $\Sigma(p_n \mid n$ even, $n \geqq 6)$ is odd, it follows from equation (3), taken modulo 2, that $\Sigma(p_n \mid n$ even, $n \geqq 4)$ is even (remark that p_2 is even), hence p_4 is odd. Apply Operation 3, followed by Operations 1, to obtain a 4-gon, a $(6+2)$-gon, monogon and triangles. The remaining even number of 4-gons is obtained by Operations 4.

The graph G^{**} obtained so far is 6-valent, connected, planar; it has precisely

p_n n-gons for all even n, $n \geqq 4$; it has no n-gons for odd n, $n \geqq 5$; it has triangles, at least three monogons (the centers of the three patches of G^*) and it has no digons. It follows, using the Lemma, that there exists a 6-valent, connected planar graph G^{***} having precisely p_n n-gons for all n, $n \geqq 4$, no digons, $p_1 + k = p_1 + \frac{1}{2}p_2$ monogons and triangles.

Every monogon of $S(S(G^{***}))$ is the center of a 1-patch (shown on the left side of Figure 8), and these 1-patches are disjoint; Operation 5 is applied in $\frac{1}{2}p_2$ disjoint parts of $S(S(G^{***}))$, yielding a 6-realization of $(p_n \mid 1 \leqq n \neq 3)$, as promised.

Case II. $p_2 = 2k + 1$ is odd. In this case it follows from equation (3), taken modulo 2, that for some even m, $m \geqq 4$, $p_m \geqq 1$ (since $\Sigma\ (p_n \mid n \text{ even}, n \geqq 2)$ is always even); let $m = 2r$.

Define a new sequence $(p_n'' \mid 1 \leqq n \neq 3)$ by $p_1'' = p_1 + k - r + 2$, $p_2'' = 0$, $p_m'' = p_m - 1$ and $p_n'' = p_n$ for all $4 \leqq n \neq m$. This sequence satisfies the condition of the previous Case I, hence there exists a 6-valent, connected planar graph G^+, which has precisely p_n'' n-gons for all n, $1 \leqq n \neq 3$.

Every monogon of $S(S(G^+))$ is the center of a 1-patch, as has been noted; apply one Operation 6 to one such monogon of $S(S(G^+))$, and convert the new 4-gon into an m-gon by $\frac{1}{2}(m-4)$ consecutive Operations 1; denote the resulting graph by G^{++}. Apply, as before, $\frac{1}{2}(p_2 - 1)$ disjoint Operations 5 on G^{++}, and we obtain a 6-valent, connected planar graph that has precisely p_n n-gons for all n, $1 \leqq n \neq 3$.

This completes the proof of Theorem 1.

Let *Operation j*, for $7 \leqq j \leqq 11$, be defined as shown in Figures 10–12.

We are ready for the proof of Theorem 2.

PROOF OF THEOREM 2. Let $(p_n \mid 1 \leqq n \neq 3)$ be an arbitrary sequence of nonnegative integers satisfying equation (5), that is,

$$2p_1 + p_2 = \sum_{n \geqq 4} (n-3)p_n.$$

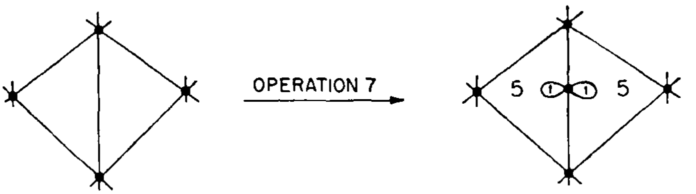

Fig. 10

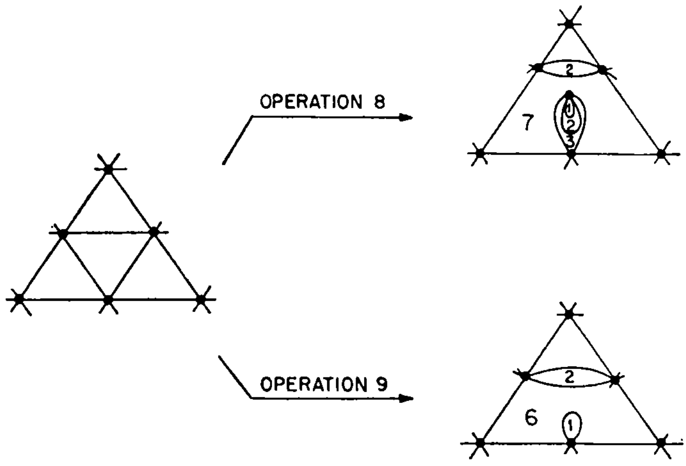

Fig. 11

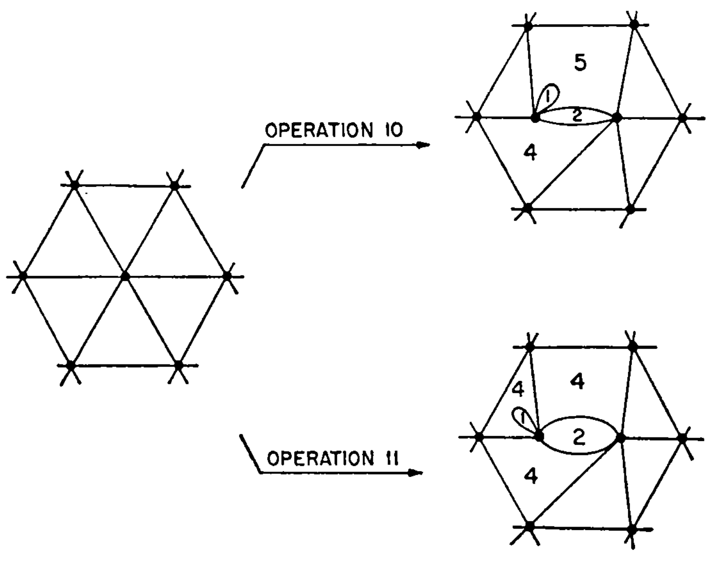

Fig. 12

Let G_1 be a 6-valent graph on the torus that triangulates the torus, and such that G_1 is so big as to allow all the needed operations in a non-overlapping manner.

Case I. p_2 is even. We divide the construction into three subcases, as follows.

i. $\Sigma(p_n \mid n \text{ even}, n \geqq 4) \geqq 1$. Starting with G_1, we proceed as in the proof of Theorem 1, Case I (where G^{**} is obtained from G^*) so as to obtain a 6-valent, connected toroidal graph G_2 that has precisely p_n n-gons for all even n, $n \geqq 4$, that has no digons and no m-gons, for all odd m, $m \geqq 5$. Unlike G^{**}, G_2 need not have at least three monogons; however, since Σ $(p_n \mid n$ even, $n \geqq 4)$ is even, it is greater than or equal to two (being at least one). hence it follows from equation (5) that G_2 has at least one monogon.

Applying the Lemma and $\frac{1}{2}p_2$ Operations 5 to $S(S(G_2))$ results in a 6-valent, connected toroidal graph G_3 that has precisely p_n n-gons for all n, $1 \leqq n \neq 3$, as promised.

ii. $\Sigma(p_n \mid n$ even, $n \geqq 4) = 0$ and $\Sigma(p_n \mid n$ odd, $n \geqq 5) \geqq 2$. Let G_4 denote the graph, obtained from G_1 by one Operation 7. Apply Operations 1 to the two monogons inside the 5-gons of G_4 so as to obtain a graph G_5 that has an r-gon and an s-gon, for odd r and s, $r, s \geqq 5$ (for which $p_r + p_s \geqq 2$, if $r \neq s$, or $p_r \geqq 2$ otherwise).

An application of the Lemma to G_5 results in a 6-valent, connected toroidal graph G_6 that has precisely p_n n-gons for all n, $n \geqq 4$, has no digons, and has $p_1 + \frac{1}{2}p_2$ monogons, in addition to 3-gons. The desired graph is obtained by $\frac{1}{2}p_2$ disjoint Operations 5, performed on $S(S(G_6))$.

iii. $\Sigma(p_n \mid n$ even, $n \geqq 4) = 0$ and $\Sigma(p_n \mid n$ odd, $n \geqq 5) \leqq 1$. If $p_n = 0$ for all odd n, $n \geqq 5$, then it follows from equation (5) that $2p_1 + p_2 = 0$, hence in this case $p_n = 0$ for all n, $1 \leqq n \neq 3$, and G_1 is a suitable toroidal 6-realization of the zero sequence.

Otherwise, let m be the only odd n, $n \geqq 5$, for which $p_n = 1$. If $m \geqq 7$, let G_7 be the 6-valent toroidal realization of the sequence $(p_7 = 1, p_1 = 2, p_n = 0$ for all $n \neq 1, 3, 7)$ as shown in Figure 13, where the torus is presented in the usual way as a rectangle with opposite sides identified in the same directions.

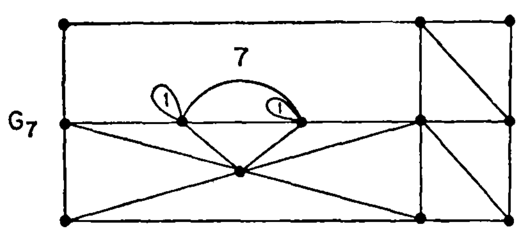

Fig. 13

Let G_8 be obtained from G_7 by $\frac{1}{2}(m-7)$ Operations 1, and let G_9 be the graph obtained from $S(S(G_8))$ by $\frac{1}{2}p_2$ Operations 5; G_9 is the desired toroidal 6-realization of the given p-vector. In a similar way, for case $m = 5$, Figure 14 presents a toroidal 6-realization G_{10} of the sequence ($p_5 = p_1 = 1, p_n = 0$ for all $n \neq 1,3,5$); a toroidal 6-realization of the sequence ($p_5 = 1, p_2 = 2, p_n = 0$ for all $n \neq 2,3,5$) is obtained from $S(S(G_{10}))$ by one Operation 5.

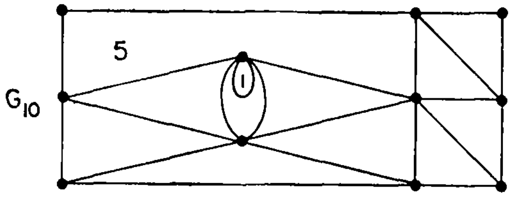

Fig. 14

This completes Case I.

Case II. p_2 is odd. It follows from equation (5), taken modulo 2, that for some even m, $m \geqq 4$, $p_m \geqq 1$. If $m \geqq 6$, then let the graph G_{11} be obtained from G_1 by Operation 9, followed by $\frac{1}{2}(m-6)$ Operations 1 on the monogon inside the n-gon, $n \geqq 6$. Using G_{11}, proceed as in the proof of Theorem 1, Case I.

If $p_n = 0$ for all even n, $n \geqq 6$, then p_4 is odd. In case $p_4 = 1$, either the sequence is ($p_4 = p_2 = 1$, $p_n = 0$ for all $n \neq 2,3,4$), which is toroidal 6-realized (shown in Figure 15, again with the usual identifications), or else for some odd r, $r \geqq 5$, $p_r \geqq 1$; apply Operation 10 to G_1 and proceed as in the previous case.

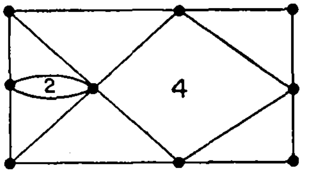

Fig. 15

If $p_4 \geqq 3$, apply Operation 11 to G_1 and proceed as in the previous case.

This completes the proof of Theorem 2.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor Branko Grünbaum for raising the questions discussed here and for closely following the derivation of their solution. The right half of Figure 5 is a simplification of our original graph, and it is due to B. Grünbaum.

REFERENCES

1. D. Barnette, E. Jucovič, and M. Trenkler, *Toroidal maps with prescribed types of vertices and faces*, Mathematika **18** (1971), 82–90.

2. G. Brunel, *Sur quelques configurations polyédrales*, Procès-Verbaux Séances Soc. Sci. Phys. et Mat. Bordeaux (1897–98), 20.

3. V. Eberhard, *Zur Morphologie der Polyeder*, Teubner, Leipzig, 1891.

4. B. Grünbaum, *Convex Polytopes*, J. Wiley, New York, 1967.

5. B. Grünbaum, *Some analogues of Eberhard's theorem on convex polytopes*, Israel J. Math. **6** (1968), 398–411.

6. B. Grünbaum, *Planar maps with prescribed types of vertices and faces*, Mathematika **16** (1969), 28–36.

7. B. Grünbaum, *Polytopes, graphs and complexes*, Bull. Amer. Math. Soc. **76** (1970), 1131–1201.

8. B. Grünbaum and J. Zaks, *The existence of certain planar maps*, to appear.

9. S. Jendrol and E. Jucovič, *On the toroidal analogue of Eberhard's theorem*, Proc. London Math. Soc. (25) **3** (1972), 385–398.

10. J. Malkevitch, *Properties of planar graphs with uniform vertex and face structure*, Ph.D. thesis, University of Wisconsin, Madison, 1969.

11. O. Ore, *The four-color Problem*, Academic Press, New York, 1967.

12. D. A. Rowland, *An extension of Eberhard's theorem*, M. Sc. thesis, University of Washington, Seattle, 1968.

13. J. Zaks, *The analogue of Eberhard's theorem for 4-valent graphs on the torus*, Israel J. Math. **9** (1971), 299–305.

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON, U. S. A.

AND

UNIVERSITY OF HAIFA
HAIFA, ISRAEL